exact values. For $b/a = 4$ the classical and the optimized Kantorovich method yield the same value of the dimensionless temperature.

Consider now the case where

$$p(x,y) = -0.2p_0\frac{y}{b} \quad (23)$$

which is the problem studied by Djukic and Atanackovic [1] and where their interesting extension of the Kantorovich method is tested. The problem is symmetric with respect to the x-variable and antisymmetric with respect to y. Accordingly the previously obtained expressions for $f(x, \gamma, \xi)$, M and N are applicable for this situation.

On the other hand the determination of $g(y)$ is straightforward.

In the case of a square shape the maximum value of $|\theta/(p_0a^2/k)|$ is 0.01138 while the approach presented in ref. [1] yields 0.01324. The present, optimized Kantorovich method yields 0.01165 which is in excellent agreement with the exact result.

Figure 2 yields a comparison of dimensionless temperature values as a function of y/a for $x/a = 0$ in the case of a square shape. The values of γ and ξ which minimize the functional are 6.7 and -0.5, respectively.

Acknowledgements—The present investigation has been sponsored by CONICET Research and Development Program (PID 3009400).

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Int. J. Heat Mass Transfer. Vol. 32, No. 3, pp. 613–616, 1989
Printed in Great Britain
0017–9310/89 \$3.00+0.00

Two temperature, two phase heat transfer in porous media: solution to linear models

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(*Received* 30 *October* 1987 *and in final form* 27 *May* 1988)

1. INTRODUCTION

TWO TEMPERATURE heat transfer in porous media and packed beds can be described by two coupled partial differential equations for the solid and fluid phases. An analytical solution for two phase heat transfer which includes time dependence, and heat source terms is important for modeling in such applications as predicting wall heat loss in reactors. Exact solutions to the coupled system are useful in that, in some cases, they require much less programming and computer time than finite difference solutions. Past solution techniques have proceeded by neglecting various terms in the equations [1, 2]. Burch *et al.* [3] introduced the idea of the interacting boundary conditions at the inlet face of the reactor for the two phase problem. Toovey and Dayan [4] presented an ingenious solution when fluid diffusion and heat capacity are neglected. What one feels is needed is a very general treatment of the heat transfer problem in porous media. In particular one contends that for problems with low Reynolds numbers it is not valid to neglect the diffusivity of the fluid or heat capacity (particularly when the fluid is very dense). In this note two new solutions to the time dependent, non-homogeneous problem are developed. The first solution utilizes an eigenfunction expansion and will be useful for problems when diffusion and convection are of the same order of magnitude. The solution is based on an eigenfunction expansion in the axial coordinate. Numerical results are presented for the case of a cylindrical reactor. The second solution is exact with no approximations and is valid for problems where source terms do not depend on position in the sample. A uniqueness proof has been developed and is presented elsewhere [2].

2. THE PROBLEM

One wishes to determine the distribution of temperatures of solid and fluid phases in a packed bed where the fluid may be moving relative to the solid. Let the temperature of the fluid and solid be T_f and T_s. The superficial velocity of the fluid is v_f. The differential equations for the heat transfer are

$$\frac{\partial T_f}{\partial t}+\bar{v}_f\cdot\nabla T_f = \nabla^2 T_f - h_f(T_f-T_s)+g_f(r,t)$$

$$\frac{\partial T_s}{\partial t} = \alpha'\nabla^2 T_s + h_s(T_f-T_s)+g_s(r,t) \quad (1)$$

h' is the dimensional heat transfer coefficient, $h_f = h'L^2/k_f$, $h_s = h'L^2\rho_f c_f/\rho_s c_s k_f$; α_f, α_s are thermal diffusivities; g_f and g_s are the non-dimensional source terms for fluid and solid, respectively. Also the non-dimensional variables t, z, T, r, v are related to dimensional quantities t', z', r', T', v' by $t' = t\tau_0$, $z' = zL$, $r' = rL$, $T' = T_0'T$, $v' = v_fL/\tau_0$ and $\tau_0 = L^2/\alpha_f$, $\alpha' = \alpha_s/\alpha_f$. The initial conditions are given by

$$T_f(t=0) = F_f(\bar{r}), \quad T_s(t=0) = F_s(\bar{r}). \quad (2)$$

The boundary conditions on the bounding surfaces are determined by an analysis of the problem of uniqueness [2]. In general one can write

$$\nabla T_v\cdot\bar{n} = H(T_v-T_0) \quad (3)$$

where ($v = $ s, f), H is a constant and T_0 is the temperature of the surroundings.

NOMENCLATURE

a	radius of reactor	s	Laplace transform variable
c_i	specific heat	T_i	temperature
C_i	constant	v_f	velocity fluid
f_n	eigenfunction	x_i	temporary variables.
F_i	initial conditions		
g_i	source terms		
h	heat transfer coefficient	Greek symbols	
H	convection coefficient	α'	diffusivity
J_0	Bessel function of order zero	β	eigenvalue
k'	thermal conductivity	η	decay factor
L	axial extent	λ_n	eigenvalue
R	temporary variable	ξ	decay factor
R_n	coefficient of eigenfunction	ρ	density.

3. SOLUTION USING AN EIGENFUNCTION EXPANSION FOR A CYLINDRICAL REACTOR

For many applications the solution to the system of equations (1) in cylindrical coordinates is quite important. Consider the reactor of radius a and axial extent (0, 1) with fluid flow only in the axial direction. One wishes to find the temperature distribution of the solid and fluid phases in the (r, z) plane. The Hankel transforms are defined by

$$\bar{\bar{T}} = \int_0^a r' J_0(\beta_m r') T(r', z, t)\, \mathrm{d}r' \tag{4}$$

where J_0 are the (normalized) Bessel functions of the first kind and β_m are eigenvalues. If equation (3) is used for the temperature, then the eigenvalues are obtained from

$$\frac{\mathrm{d}J_0(\beta_m a)}{\mathrm{d}r} = HJ_0(\beta_m a). \tag{5}$$

The spatial transform is defined by

$$\bar{T} = \int_0^1 \exp(-v_f z') f_n(z') T(r, z', t)\, \mathrm{d}z' \tag{6}$$

where the eigenfunctions for the axial coordinate satisfy

$$\mathrm{d}^2 f_n/\mathrm{d}z^2 - v_f \mathrm{d}f_n/\mathrm{d}z = -\lambda_n^2 f_n \tag{7}$$

where λ_n are the eigenvalues of the operator. The boundary conditions are given by

$$\mathrm{d}f_n/\mathrm{d}z = Hf_n. \tag{8}$$

The inverse transform is defined as

$$T = \sum_n 2\bar{T}_n f_n(z). \tag{9}$$

Therefore, equations (7) and (8) constitute a Sturm–Liouville boundary value problem. The eigenfunctions f_n form a complete set and the eigenvalues λ_n^2 are all real numbers and not more than a finite number of them are negative. The eigenfunctions are orthogonal over the interval [0, 1] with respect to the weight function $\exp(-v_f z)$. The norm is defined as

$$\|f_n\|^2 = \int_0^1 \exp(-v_f z) f_n^2(z)\, \mathrm{d}z. \tag{10}$$

In the following analysis it is assumed that the eigenfunctions are normalized. The general solution to equation (7) can be written as

$$f_n(z) = (2)^{1/2} \exp(v_f z/2)\, [\cos(\gamma z) + R_n \sin(\gamma z)] \tag{11}$$

where R_n and γ are constants to be determined by equation (8). The constant γ is determined from, $\lambda_n^2 = \gamma^2 + v_f^2/4$

$$\frac{[(v_f/2 - H) - \gamma \tan \gamma L]}{(-v_f/2 + H) \tan \gamma L - \gamma} = \left(\frac{-v_f}{2\gamma} + \frac{H}{\gamma}\right) \tag{12}$$

and the constant R_n is determined from

$$R_n = (H - v_f/2)/\gamma. \tag{13}$$

If one specifies temperatures at $z = 0, 1$, then the eigenfunctions are $\sin(n\pi z)\exp(v_f z/2)$ and the eigenvalues are $\lambda_n^2 = (n\pi)^2 + v_f^2/4$. After taking the Hankel, Laplace and spatial transforms of equations (1), the following constants are obtained:

$$C_1 = -[\lambda_n^2 + \beta_m^2 + h_f], \quad C_2 = h_f$$

$$C_3 = \left\{ \bar{\bar{\bar{g}}}_f - \bar{\bar{T}}_f \frac{\mathrm{d}}{\mathrm{d}z}[f_n \exp(-v_f z)]|_0^1 + a\left[\frac{\partial \bar{T}_f}{\partial r}(r = a) J_0(\beta_m a) - \bar{T}_f(r = a)\frac{\mathrm{d}J_0(\beta_m a)}{\mathrm{d}r}\right]\right\}$$

$$C_4 = -\left[\alpha'\left[(n\pi)^2 - \left(\frac{v_f}{2}\right)^2\right] + \alpha'\beta_m^2 + h_s\right], \quad C_5 = h_s$$

$$C_6 = \left[\bar{\bar{\bar{g}}}_s + \alpha' a\left[\frac{\partial \bar{T}_s}{\partial r}(r = a) J_0(\beta_m a) - \frac{\mathrm{d}J_0(\beta_m a)}{\mathrm{d}r}\bar{T}_s(r = a)\right] - \alpha' \bar{\bar{T}}_s \frac{\mathrm{d}}{\mathrm{d}z}[f_n \exp(-v_f z)]|_0^1 + C_7\right]$$

$$C_7 = \alpha' \frac{v_f n\pi}{2} \int_0^1 \bar{\bar{T}}_s \cos(n\pi z) \exp(-v_f z)\, \mathrm{d}z. \tag{14}$$

Here $\bar{\bar{\bar{\ }}}$ denotes combined Laplace, axial and Hankel transforms. The solution for the temperatures is then obtained by inverting the Laplace, axial, and Hankel transforms

$$T_v = \sum_m \sum_n J_0(\beta_m r) f_n(z) \left\{ \frac{A_{vmn} \exp(-B_{mn} t/2) \sinh G_{mn} t}{G_{mn}} + \tilde{F}_v \left[\exp(-B_{mn} t/2) \cosh G_{mn} t - \frac{B_{mn} \exp(-B_{mn} t/2) \sinh G_{mn} t}{2G_{mn}} \right] \right.$$

$$+\int_0^t D_{vmn}(\tau)\exp(B_{mn}(t-\tau)/2)\cosh G_{mn}(t-\tau)\,d\tau$$

$$+\int_0^t\left[\frac{E_{vmn}(\tau)}{G_{mn}}-\frac{B_{mn}D_{vmn}(\tau)}{2G_{mn}}\right]\exp(-B_{mn}(t-\tau)/2)$$

$$\left.\times\sinh G_{mn}(t-\tau)\,d\tau\right\} \quad (15)$$

where (v = s, f) and

$$A_{smn}=C_5\tilde{F}_f-C_1\tilde{F}_s, \qquad A_{fmn}=C_2\tilde{F}_s-C_4\tilde{F}_f$$
$$B_{mn}=-(C_1+C_4), \qquad C_{mn}=C_1C_4-C_2C_5$$
$$D_{fmn}(\tau)=C_3(\tau), \qquad E_{fmn}=C_2C_6(\tau)-C_4C_3(\tau)$$
$$D_{smn}(\tau)=C_6(\tau), \qquad E_{smn}=C_5C_3(\tau)-C_1C_6(\tau)$$
$$G_{mn}=\left(\frac{B_{mn}^2}{4}-C_{mn}\right)^{1/2}$$

where $\sim$ denotes combined Hankel and axial transforms. To a very excellent approximation one can assume $C_7=0$. This approximation is valid when $(v_f/2)^2-(n\pi)^2 \gg v_f n\pi$ which amounts to stating $v_f \gg 3.41n\pi$ or $v_f \ll 0.6n\pi$. Note that these conditions are valid for most problems for small n (which is all that is necessary since at large n the system has converged).

4. EXACT SOLUTION WHEN THE SOURCE TERMS DO NOT DEPEND ON AXIAL COORDINATE

An exact solution with no approximations is possible for equations (1) when the sources $g(t)$ depend at most on time. If one writes the solution to equations (1) as

$$\langle T_v\rangle = T_v+T_{0v}(t) \quad [v=s,f], \quad \langle T_v(t=0)\rangle = 0 \quad (16)$$

where T_{0v} is a function only of time and is given by

$$T_{0s}=\int_0^t \exp[-(h_s+h_f)(t-\tau)] \times\left[\int_0^\tau [h_s g_f(\theta)+h_f g_s(\theta)]\,d\theta+g_s(\tau)\right]d\tau \quad (17)$$

$$T_{0f}=-\frac{h_f}{h_s}T_{0s}+\int_0^t\left(g_f(\theta)+\frac{h_f g_s(\theta)}{h_s}\right)d\theta. \quad (18)$$

Then T_v is the solution to the homogeneous forms of equations (1). The solution to equations (1) can be found by taking Laplace transforms and substituting the resultant equations into each other to decouple

$$\frac{d^4T_f}{dz^4}-\frac{v_f d^3T_f}{dz^3}-[(s+h_s)/\alpha'+(s+h_f)]\frac{d^2T_f}{dz^2} +\frac{(s+h_s)}{\alpha'}\frac{dT_f}{dz}-\frac{h_s h_f}{\alpha'}T_f=0 \quad (19)$$

where s is the Laplace transform variable. Then the solid temperature can be found from

$$T_s=(s/h_f+1)T_f+\frac{v_f}{h_f}\frac{dT_f}{dz}-\frac{d^2T_f}{h_f dz^2}. \quad (20)$$

To solve these equations one assumes an exponential solution $T_f(z,s)=Q\exp(\eta z)$, and substitute into equation (19) to yield a quartic equation which can be solved in a straightforward manner. If one defines the following constants one can write the solution

$$X_1=-v_f/\alpha_f, \quad X_2=-(s+h_s)/\alpha_s-(s+h_f)/\alpha_f,$$
$$X_3=v_f(s+h_s)/\alpha_f\alpha_s,$$
$$X_4=h_sh_f/((s/h_s+1)(s/h_f+1)-1)\alpha_s\alpha_f, \quad X_5=-X_2,$$
$$X_6=(X_1X_3-4X_4), \quad X_7=-X_1^2X_4+4X_2X_4-X_3^2,$$
$$X_8=\frac{1}{3}(3X_6-X_5^2), \quad X_9=\frac{1}{27}[2X_5^3-9X_5X_6+27X_7],$$
$$Y=\frac{X_5}{3}+[-X_9/2+(X_9^2/4+X_8^3)^{1/2}]^{1/3} +\left[\frac{-X_9}{2}-(X_9^2/4+X_8^3/27)^{1/2}\right]^{1/3},$$
$$R=[X_1^2/4-X_2+Y]^{1/2},$$
$$D=\left[\frac{3X_1^2}{4}-R^2-2X_2+\frac{4X_1X_2-8X_3-X_1^3}{4R}\right]^{1/2}.$$
$$E=\left[\frac{3X_1^2}{4}-R^2-2X_2-\frac{4X_1X_2-8X_3-X_1^3}{4R}\right]^{1/2}.$$

The roots are then given by

$$\eta=-X_1/4+R/2\pm D/2, \quad -X_1/4-R/2\pm E/2.$$

So the general solution can be written as

$$T_f=A'\exp(\eta_1 z)+B'\exp(\eta_2 z)+C'\exp(\eta_3 z)+D'\exp(\eta_4 z). \quad (21)$$

T_s is found by substituting equation (21) into equation (20), and constants A', B', C', D' can be found by applying the boundary conditions of equation (3) at $z=0,1$. For time independent problems, equation (21) is the closed form solution, whereas for time dependence it is necessary to take the inverse Laplace transform. Solutions to equations (21) are plotted in Figs. 3 and 4 for various v_f and h.

5. NUMERICAL RESULTS

In order to check the solution (equation (15)) a numerical example was generated on a computer for a packed bed immersed in a fluid that was assumed infinite in radial extent. The axial dimensions were [0, 0.1]. The bed was maintained at $z=0$ at a temperature of $T=1.0$, and at the bottom ($z=1$) the temperature was maintained at $T=0$. Note that in this position one utilizes interacting boundary conditions at the endpoints. The problem then is to obtain both the fluid and solid temperatures as a function of time. The tem-

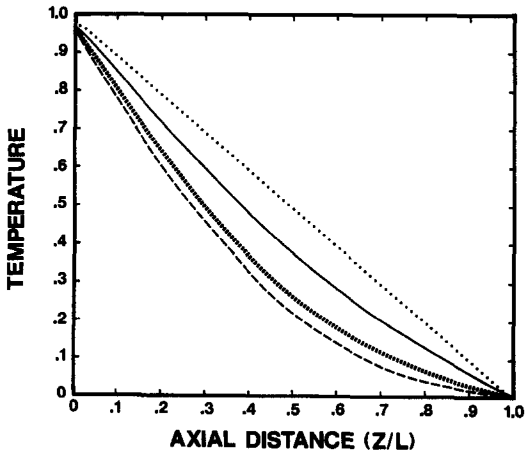

FIG. 1. Solution to equation (15) with $t'=1000$ s, $v_f=0$, $\xi=\infty$, $\alpha_f=1\times10^{-4}\,m^2\,s^{-1}$, $\alpha_s=2\times10^{-6}\,m^2\,s^{-1}$, fluid temperature with $h'=0$ (dotted line), with $h'=100\,W\,°C^{-1}\,m^{-3}$ (solid line), solid temperature with $h'=0$ (thin dashed line) and for $h'=100\,W\,°C^{-1}\,m^{-3}$ (heavy dashed line).

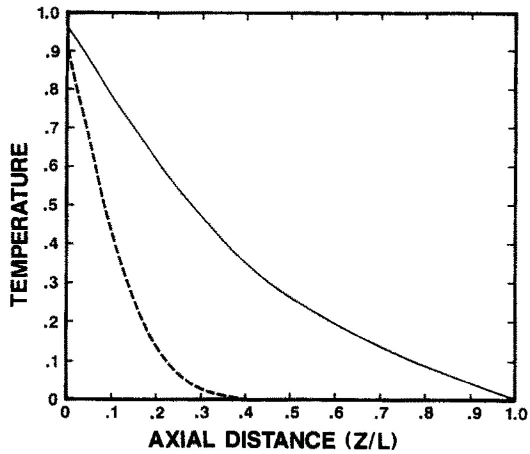

FIG. 2. Solution to equation (15) with $t = 100$ s, $v_f = 0$, $\xi = 0.1$ s^{-1}, $\alpha_f = 1 \times 10^{-4}$ m^2 s^{-1}, $\alpha_s = 2 \times 10^{-6}$ m^2 s^{-1}, fluid temperature with $h' = 0$ (dotted line), with $h' = 100$ W °C^{-1} m^{-3} (solid line), solid temperature with $h' = 0$ (thin dashed line) and for $h' = 100$ W °C^{-1} m^{-3} (heavy dashed line).

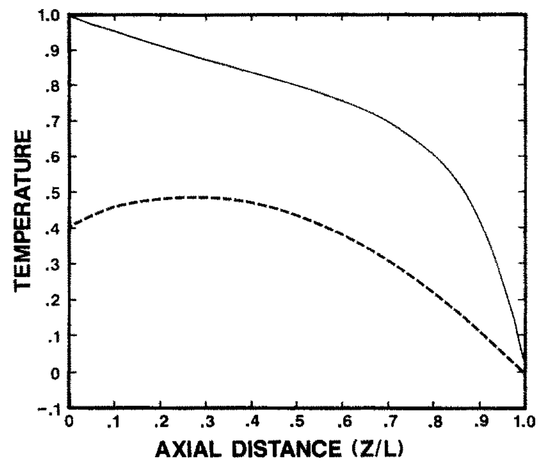

FIG. 4. The solution to equation (21) for $v_f = 0.01$ m s^{-1}, $h' = 100.0$ W °C^{-1} m^{-3}, $\alpha_s = 1 \times 10^{-6}$ m^2 s^{-1}, $\alpha_f = 1 \times 10^{-4}$ m^2 s^{-1}, $T_s'(z = 0) = 0.4$°C, $T_s'(z = 1) = 0$, $T_f'(z = 0) =$ 1°C, $T_f'(z = 0) = 0$.

perature of the solid phase at the entrance ($z = 0$) is assumed to warm as time increases due to the interaction of the solid phase with the warmer fluid phase. One assumes the following temperature for the solid at the inlet (that is one assumes that the solid temperature at the input point warms due to the interaction with the warmer fluid)

$$T_s(z = 0) = T_{f0}[1 - \exp(-\xi t)] \qquad (22)$$

where ξ is a decay constant which is set by experiment and T_{f0} is the inlet gas temperature. From Figs. 1 and 2 it can be

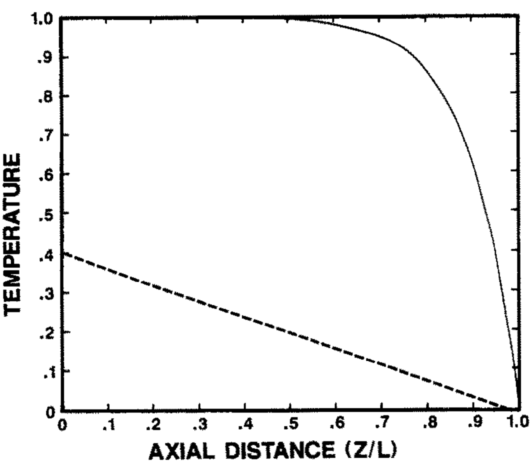

FIG. 3. The solution to equation (21) for $v_f = 0$, $h' =$ 100.0 W °C^{-1} m^{-3}, $\alpha_s = 1 \times 10^{-6}$ m^2 s^{-1}, $\alpha_f = 1 \times 10^{-4}$ m^2 s^{-1}, $T_s'(z = 0) = 0.4$°C, $T_s'(z = 1) = 0$, $T_f'(z = 0) = 1$°C, $T_f'(z = 1) = 0$.

seen that the effect of non-zero h' for this case is to warm the solid phase and cool the fluid. The net effect of non-zero h' with zero velocity is to produce equilibrium of the system. For cases with fluid flow thermal equilibrium is not necessarily the result. In Figs. 3 and 4 the steady-state solutions to equation (21) are plotted for various values of v_f and h'. For this example one specifies the temperatures of the fluid and solid at the endpoints. In Fig. 3 the velocity is nonzero and $h' = 0$ so one can see that the fluid temperature is warmer. In Fig. 4 both v_f and h' are nonzero and one can see that the fluid cools and the solid warms up due to interaction.

Acknowledgement—Acknowledgement is made to the donors of the Petroleum Research Fund administered by the ACS, for partial support of this research Grant No. ACS-PRF No. 20304-G2 and DOE Grant No. DE-AC20-85LC-11060.

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